Critical Threshold for SIRS Model on Small World Networks

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Abstract

We study the phase transition from the persistence phase to the extinction phase for the SIRS (susceptible/infected/refractory/susceptible) model of diseases spreading on small world network. We show the effects of all the parameters associated with this model on small world network and we create the full phase space. The results we obtained are consistent with those obtained in Ref. [7] in terms of the existence of a phase transition from a fluctuating endemic state to self-sustained oscillations in the size of the infected subpopulation at a finite value of the disorder of the network. Our results also assert that, transition specifically occurs where the average clusterization shifts from high to low. The effect of clustering coefficient on SIRS model on the networks can be understood from the results obtained in Ref. [9], which indicates the importance of existing the loops in the network, in order to the disease to spread frequently throughout the nodes of network. Where, clusters tend to spread infection among close-knit neighborhoods. Hence, when the loops are high inside the network, the reinfection occurs in the network at many places and at different times, which looks like as a kind of randomness in occurring the second period of infection. Whereas when the number of loops are low, reinfection occurs at specific places and times on the network, which looks like as a kind of regularity in occurring the second period of infection.

Keywords: Critical Threshold, SIRS Model, Networks.

Introduction

In the modelling of many interacting particles on the networks, the effect of the networks structure on the properties of dynamical systems defined on such networks has been attracted a lot of attention recently. Researchers from different fields ranging from neurodynamics and ecology to social sciences have been extensively working in this area. In small world networks, one starts with a ring of N nodes, in which each node connected to its k nearest neighbors on either side. Then each link from a site to its nearest neighbor is reconnected to another randomly chosen lattice site with probability p. This model is proposed to mimic real life situations in which non-local connections exist along with predominantly local.

SIRS for epidemic model on the networks is defined as follows, for a lattice of N nodes in which each node connected to its k neighbors. The nodes can exist in one stage of three states, susceptible S, infected I and refractory R. Susceptible node can pass to the infected state through contagion by an infected one. Infected node pass to the refractory state after an infection time $\tau_I$. Refractory nodes return to the susceptible state after a recovery time $\tau_R$. The contagion is possible only during the S phase and only by I node. During the R phase, the nodes are immune and do not infect.
Kuperman and Abramson\textsuperscript{7} studied SIRS model on
small world network with the following assumption: The susceptible node at time \( t \), will be
infected at time \( t + 1 \) with probability proportional
to the fraction of infected nodes in its
neighborhood. In other words, if \( \tau_i(t) = 0 \), then
\[ \tau_i(t + 1) = 1 \]
with the probability \( \lambda_i = \frac{k_{\text{inf}}}{k_i} \)
where \( k_i \) are total number of neighbors of site \( i \), of
which \( k_{\text{inf}} \) are infected. With probability \( 1 - \lambda_i \),
susceptible node does not change state. The
dynamics for the infected node is deterministic.
The infected node slowly become refractory and
then eventually become susceptible again. For the
values of \( k = 3, \tau_i = 4 \) and \( \tau_R = 5 \), they found
that, for the more ordered systems, there is a
fluctuating endemic state of low infection.
However, at a finite value of the disorder of the
network, they get a transition to self-sustained
oscillations in the size of the infected subpopulation.
In this work, we illustrate the effect of
all parameters associated with this system on
small world network.

Simulation Results

Here we study the effect of infection time \( \tau_i \) on the
steady state of the original model of Kuperman and
Abramson\textsuperscript{7}, as the long range connection \( p \) is
changed, for the case when \( \tau_R > \tau_i \). We set the
values of other parameters as that in original model
unless we state different.

\[ \text{Fig. 1: Density of infectious nodes as function of time for different values of } \tau_i \text{ and } \tau_R, \text{ as shown in the legend. Other parameters are: } N = 10^4, k = 3, p = 0.05 \text{ and } n_{\text{inf}}(0) = \text{minimum}. \]

Fig. 1 shows the effect of the increasing on the
value of the infection time on the average value of
the density of infected nodes \( n_{\text{inf}}(t) \) of this model
on regular a one dimensional lattice. In that figure,
we plot the density of infected nodes as function of
time at different values of \( \tau_i \) and \( \tau_R \). For each curve
on the figure, we set the value of \( \Delta \tau = \tau_R - \tau_i \) to
be minimum, i. e. \( \Delta \tau = 1 \). As the figure shows, the
density of the infected nodes during the first
infection period increases as the value of \( \tau_i \)
increases and that density reaches the maximum
value when \( \tau_i = 10 \).

It is clear from Fig.1 that, when \( \tau_i = 10 \), and after
a short time, all the nodes on the network become
sick (where the density of infected nodes initially
is \( n_{\text{inf}}(0) = 0.1 \), and after \( 11 \) time-steps it
becomes \( n_{\text{inf}}(11) = 0.90 \) during the first
infection period, hence system goes to the infection
free state. For this case, it is evident that, any
infectious node on the network infects all of its
neighbors during its first infection period. So and
according to Ref. \[9\] this system reaches extinction
state, where all the nodes on the network become
susceptible, and also the probability of getting two
neighbors which were being infected with time
difference \( t > \tau_R \) will be zero (see Fig. 2, when
\( \tau_i = 10 \)). Therefor, for this case, all the nodes on
the network pass only through one infection period,
and system goes to an absorbing state.

However, for the case when \( \tau_i = 6 \) and \( \tau_j = 8 \) as
Fig. 1 shows, the density of infected nodes
approaches the maximum value during the first
infection period, while there are a significant
numbers of nodes still unaffected. That means on
the average, each infected node on the network
does not infect all of its neighbors during its
infection period. Hence, those uninfected nodes
previously, there is a possibility to become lately
infected by their second or third etc. infected
neighbor. Thus in this case, the probability to get
two neighbors on the network with \( t > \tau_R \) is
possible (see Fig. 2, when \( \tau_i = 6 \) and \( \tau_j = 8 \)).
Therefore, this behavior prevents the system from
falling to an absorbing state from the first infection
stage\textsuperscript{9}.

For the same values of parameters in Fig. 1, we
represent in Fig. 2 the density of pairs of neighbors
which they have been infected with a time
difference \( t > \tau_R \), as function of time. In that
calculation, we consider only the nodes, which are
in the states I and R. It is clear that, the density of
pairs of neighbors, which they have been, infected
with a time difference \( t > \tau_R \) decreases as the
value of \( \tau_i \) increases. Fig. 2 shows that, when \( \tau_i = 10 \),
the density of pairs of neighbors with \( t > \tau_R \)
goes to zero. However, when \( \tau_i < 10 \) there are
significant numbers of pairs of neighbors with \( t > \tau_R \).

\[ \text{http://rujms.alraziuni.edu.ye/} \]

For completeness, we examine the model as the value of \( \Delta t = 3 \). In general we find that, for any values of \( \tau_I \) and \( \tau_R \) which satisfy the condition \( \tau_I < \tau_R \), the system evolves to an extinction when \( \tau_I + \tau_R = 21 \).

Situation becomes more complicated on the small world network where, the nodes have different numbers of nearest neighbors \( k_i \). There are nodes become heavily connected, such nodes will need less time on the average until they become infected. However, there are some other nodes become less connected, which means on the average they will need longer time until they become infected. We have performed extensive numerical simulations at different values of \( p \) ranging from \([0.01 - 1.0]\). Interestingly we find that, for any value of \( p \), the system reaches an extinction state when \( \tau_I + \tau_R \approx 21 \), in behavior similar to what happens on the regular lattice. This result is expected where, small world network of Watts and Strogatz which we use in our network has on average a fixed connectivity \( k = 2k^c \) for any values of the disorder parameter \( p \).

Finally, we study the effects of the parameters \( \tau_I \), \( \tau_R \) and \( p \) on the steady state of this model. Fig. 3 shows, the density of infected nodes as function of time at different values of \( \tau_I \) and \( \tau_R \). Fig. 3a, shows three time series of \( n_{\text{inf}}(t) \) when the value of the disorder parameter \( p \) is \( p = 0.2 \). In these curves, we fix the value of \( \tau_I = 6 \), and \( \tau_R \) takes the values \( \tau_R = 9 \) (bottom), \( 11 \) (middle), and \( 12 \) (top). It is evident that, as the value of \( \tau_R \) increases, the system crosses from the fluctuating endemic state (when \( \tau_I = 9 \)) to an oscillatory state (when \( \tau_I = 11 \)). Even if the amplitude of oscillation is slightly small, but it is almost periodic with a very well defined period. Fig. 3b shows two time series of \( n_{\text{inf}}(t) \) when the value of the disorder parameter \( p \) is \( p = 0.8 \). In the two curves, we fix the value of \( \tau_I = 6 \), and \( \tau_R \) takes the values \( \tau_R = 7 \) (top), and \( 10 \) (bottom). It is clear in this case, the large amplitude self-sustained oscillation is developed.

In Fig. 4, we create the phase space of the SIRS model at several values of \( p \). For each value of \( p \), we study the system at various values of \( \tau_I \) and \( \tau_R \). We find that, when value of the disorder parameter \( p \) is bigger than \( 0.14 \), we can distinguish between three phases: a susceptible-absorbing phase, a self-sustained oscillation phase and a fluctuating endemic phase. Whereas, when \( p < 0.14 \) we observe only two phases, a susceptible-absorbing phase and a fluctuating endemic phase.
In Fig. 4, for the case when $p = 0.1$ the regions II+III+IV (the regions under the black solid line) are corresponding to the fluctuating endemic phase, whereas the region I (the region upper the black solid line) is corresponding to an absorbing phase. The black solid line is the critical line that separates the absorbing phase from the coexistence stable phase. However, when $p = 0.2$, the model shows the three phases: the susceptible-absorbing phase is the region I (the region upper the black solid line), self-sustained oscillation phase is corresponding to the region II (the region enclosed by the solid line and the dotted curve), and the regions III+IV are corresponding to the fluctuating endemic phase (the remaining region under the solid line and the dotted curve). The dotted curve is the critical curve that separates the oscillation phase from the absorbing phase. When $p = 0.8$, the oscillation phase is corresponding to the regions II and III (the region enclosed by the solid line and the dashed curve). It is clear that, the region corresponding to the oscillation phase shrinks as the value of $p$ decreases and becomes wider as the value of $p$ increases. Here, we can infer that, the critical value of $p$, which separates the oscillation phase from the fluctuating endemic phase, should be in between $0.1 < p < 0.2$. For best estimate, the critical point is approaching the value $p_c = 0.14 \pm 0.02$. at the values $\tau_f = 7$ and $\tau_R = 13$.

The value of $p_c$ we find here approximately is the value of $p$ where the average clusterization shifts from high to low as mentioned in Ref. 7. We support that conclusion with the following argument. It had been proved in Ref. 9 that, the clustering coefficient will play an important role in the SIRS model, where existence the loops on the network is necessary in order to the disease to spread frequently throughout the nodes of the networks. Whereas, clusters tend to spread infection among close-knit neighborhoods. We speculate that, whenever the value of clustering coefficient is high the next period of infection will happen at many places on the network and at any time, which will look like as a kind of randomness (in space and time) in the next generation of infection. However, when the clustering coefficient becomes lower, which means the number of triangular loops on the network also will become lower, the reinfection will be localized where those loops exist, consequently the next period of infection (on the average) will happen at specific place and time on the network. This behaviour becomes more apparent as the value of clustering coefficient becomes smaller at higher values of $p$, where the periodicity of oscillation becomes smoother.

Here, we point out to that, Phase transition at specified randomness values of small world network has been observed also in many other systems. For example, a propagation of rumors on small world networks, the introduction of shortcuts enhances the network synchronizability in a system of coupled oscillatory elements, also in self-sustained activity of excitable neurons, the introduction of shortcuts changes the probability of failure from 0 to 1 over a narrow range in $p$. In Ising model, the addition of shortcuts induces a finite-temperature phase transition even in the one-dimensional, and the introduction of unidirectional shortcuts can change the second-order phase transition in the two dimensional Ising model into a first-order one.

**Conclusion**

We have studied the spreading of infectious diseases for the SIRS model on small world network. We examine the effects of all parameters related to this model on its steady state. We find that, when the disorder parameter is $p > 0.14$, we can distinguish between three phases: a susceptible-absorbing phase, a self-sustained oscillation phase and a fluctuating endemic phase.
However when $p < 0.14$ we find only two phases: a susceptible-absorbing phase and a fluctuating endemic phase. For best estimate, $p = 0.14 \pm 0.02$ is the critical value, which separates the oscillation phase from the fluctuating endemic phase for this model on small world network.

References